Simple procedures for obtaining viscosity/shear rate data from a parallel disc viscometer

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(Received 21 April 1986; revised 16 June 1986)

New methods are presented for obtaining viscosity/shear rate data with a parallel disc viscometer, from measurements of viscous torque as a function of disc separation c and angular velocity Ω. In a previous paper it was shown that the rim shear stress P_R can be derived from an equation of the form

$$T+\frac{y}{3}\frac{dT}{dy}=\frac{2}{3}\pi R^3 P_R$$

where T is the measured torque, R the disc radius and $y=\Omega/c$. The present paper makes use of the same equation but presents a modified procedure which avoids the need to evaluate the derivative dT/dy. A first order approximation enables viscosity/shear rate data to be derived in the same way as for a cone and plate viscometer. It is shown that for shear-thinning fluids the error involved is $\leqslant 1\%$. For shear-thickening fluids, or where higher accuracy is demanded, a second order method is introduced. This enables the error to be reduced by at least an order of magnitude.

(Keywords: viscosity; shear; polyisobutylene; parallel disc viscometer)

INTRODUCTION

In the cone and plate viscometer the shear rate (G) is independent of the radius (r) and is given by the relation

$$G=\Omega/\alpha \quad (1)$$

where Ω is the angular velocity and α is the cone angle. This situation of uniform shear rate is a unique feature of cone and plate geometry. It results in a shear stress (P) which is also independent of r and is related to the measured torque (T) by the simple equation

$$P=3T/(2\pi R^3) \quad (2)$$

It is thus the ideal geometry for measurements on non-Newtonian fluids, but with disperse systems there can be problems associated with the narrow gap between the platens[1]. A previous paper[2] presented a number of techniques employing modifications of the geometry to achieve wider platen separation. Three configurations were investigated; displaced cone and plate, annular cone and plate and parallel disc. The annular cone and plate maintains the condition of uniform shear rate and produces a data treatment similar to the conventional cone and plate system. However, with the other systems the shear rate is not uniform and in consequence the shear stress is an unknown function of the radius. The basic problem is then the computation of shear stress from torque measurements.

In the configuration where the cone is displaced from its theoretical position of contact to give a clearance c between the apex of the cone and the plate it has been shown[2] that

$$T-\frac{c}{3}\frac{dT}{dc}=\frac{2}{3}\pi R^3 P_R \quad (3)$$

enabling the rim stress P_R to be calculated from measurements of T as a function of c. The parallel disc system can be regarded as a displaced cone configuration with zero cone angle and equation (3) is still applicable. It is, however, simpler and has certain advantages. Thus the shear rates depend on Ω/c, enabling the equation to be written in the form

$$T+\frac{y}{3}\frac{dT}{dy}=\frac{2}{3}\pi R^3 P_R \quad (4)$$

with

$$y=\Omega/c$$

In fact, the equation is also true if $y=\Omega$ or $y=1/c$. Equation (4) enables P_R to be evaluated from measurements of T as a function of y, or indeed of Ω or c separately. dT/dy can be evaluated graphically or by computer techniques. The present paper introduces additional techniques with parallel discs which avoid the error due to numerical calculation of derivatives.

CORRECTED SHEAR RATE METHOD

With conventional cone and plate geometry we have the equation

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$$T = \frac{2}{3}\pi R^3 P_R \tag{5}$$

and, comparing equations (4) and (5), $(y/3)\mathrm{d}T/\mathrm{d}y$ may be regarded as a correction for non-uniformity of shear rate, giving a corrected torque T', represented by the equation

$$T' = T + \frac{1}{3}y\frac{\mathrm{d}T}{\mathrm{d}y} \tag{6}$$

For the simple case of a Newtonian fluid sheared between parallel discs we have

$$T = \int_0^R 2\pi r^2 P_r \,\mathrm{d}r$$

with

$$P_r = \eta G_r = \eta \Omega r / c$$

Hence

$$T = \tfrac{1}{2}\pi R^4 \eta \Omega / c$$

Thus, T is proportional to y. With a non-Newtonian fluid the relation between T and y is non-linear, but it is assumed that the T/y curve can be approximated by a series of straight lines each extending over a range from y to $4y/3$, that is, that over this range $\mathrm{d}T/\mathrm{d}y$ is essentially constant. If then we start at a point (y,T) on the curve and traverse the linear section until y increases by $y/3$, T will increase by $(y/3)\mathrm{d}T/\mathrm{d}y$ to give a value of $T+(y/3)\mathrm{d}T/\mathrm{d}y$ and from equation (6) the coordinates are now $(4y/3, T')$. On the basis of equation (4) P_R is calculated from the corrected torque T' which approximates to the value at $4y/3$. In the corrected shear rate method the procedure is reversed. The shear stress is calculated from the uncorrected torque T using $P = 3T/(2\pi R^3)$ and the corresponding shear rate is $3\Omega R/(4c)$.

MATHEMATICAL INTERPRETATION

We have so far given a physical justification of the method. We can define the problem mathematically as follows. We wish to find $P_R(y)$ given the experimental function $T(y)$ and

$$T(y) + \frac{y}{3}\frac{\mathrm{d}T(y)}{\mathrm{d}y} = \frac{2\pi}{3}R^3 P_R(y)$$

We propose to let

$$T(\lambda y) = T + \frac{y}{3}\frac{\mathrm{d}T}{\mathrm{d}y} \tag{7}$$

and we shall take a value of λ which gives us a best approximation. Clearly the quality of the approximation depends on the nature of the function $T(y)$ and in turn on $P(G)$. The method does not assume the form of $P(G)$ but we look at some examples to find out how well our suggested value of 4/3 for λ fits the data and to get some idea of the accuracy of the method.

The Bingham fluid

In a flowing Bingham fluid the shear stress P is related to the shear rate G by

$$P = A + BG$$

Computing T we find it is a linear function of y, say

$$T = \alpha + \beta y$$

Inserting this function of T into equation (7) we find that the equation is satisfied for all values of y if

$$\lambda = 4/3$$

This, of course, is the justification for the use of 4/3 since a Bingham fluid reduces to a Newtonian liquid when A is zero.

The power law liquid

The shear stress and shear rate of this material are related by

$$P = AG^n$$

which leads to

$$T = \alpha y^n$$

Inserting in equation (7) we find the equation is satisfied for all y if

$$1 + n/3 = \lambda^n$$

Thus the best choice of λ depends on n. If n is one, a Newtonian liquid, then λ is 4/3. In general, the relative error E in $P_R(y)$ will be

$$E = \left[T(\lambda y) - \left(T + \frac{y}{3}\frac{\mathrm{d}T}{\mathrm{d}y}\right)\right] \Big/ \left[T + \frac{y}{3}\frac{\mathrm{d}T}{\mathrm{d}y}\right] \tag{8}$$

which on calculation is found to be

$$E = 3\lambda^n/(3+n) - 1 \tag{9}$$

A plot of this function for various values of n and λ is given in *Figure 1*. If we take λ as 4/3 we can look for the maximum error. If $0 < n < 1$, a shear thinning liquid, we find

$$E_{max} = -1.028 \times 10^{-2}$$

with $n = 0.47606$. The use of $\lambda = 4/3$ is not good for a shear thickening liquid, a material with $n > 1$. A better choice for λ would be

$$\lambda = 1.3048$$

if n is known to lie in the range $1 < n < 2$. The experimental values of T will give an indication of the value of n since, if P is proportional to G^n, then T is proportional to y^n.

The Casson equation

The Casson equation, originally proposed for pigment suspensions[3], takes the form

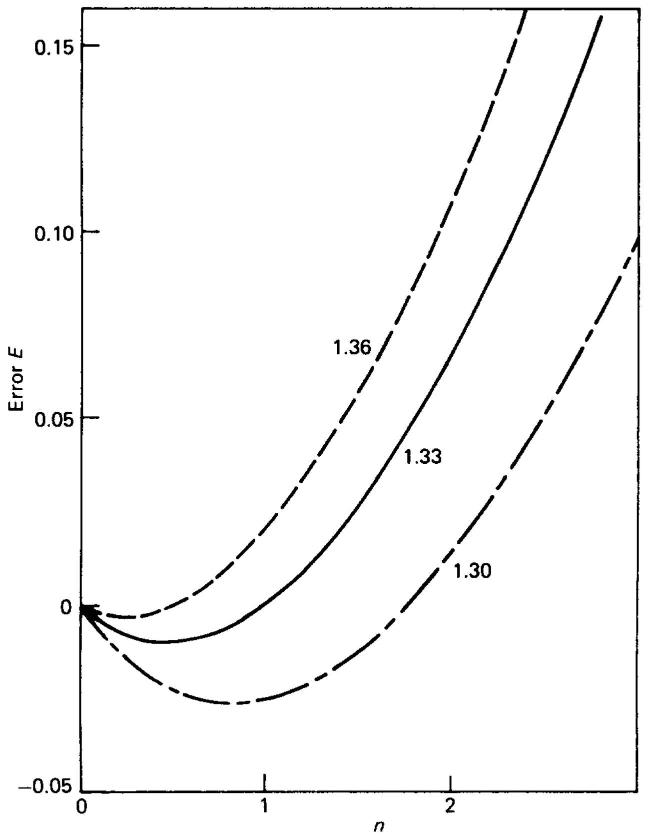

Figure 1 Error E as a function of power law index n and λ. First order methods. (— — —) $\lambda=1.30$; (———) $\lambda=4/3$; (– – –) $\lambda=1.36$

$$P^{1/2}=k_0+k_1G^{1/2}$$

which leads to the functional form of T given by

$$T=\alpha+\beta y^{1/2}+\gamma y$$

where α, β and γ are known functions of k_0, k_1 and R. Inserting this functional form of T in equation (7) we obtain

$$\alpha+\frac{7}{6}\beta y^{1/2}+\frac{4}{3}\gamma y=\alpha+\beta\lambda^{1/2}y^{1/2}+\gamma\lambda y$$

There is no way of satisfying this equation for all values of y by a special choice of λ. If we choose λ as 4/3 then we find the error E given by

$$E=\left[\alpha+\left(\frac{4}{3}\right)^{1/2}\beta y^{1/2}+\frac{4}{3}\gamma y\right]\Bigg/\left[\alpha+\frac{7}{6}\beta y^{1/2}+\frac{4}{3}\gamma y\right]-1 \quad (10)$$

with α, β and γ positive there is a maximum error. No matter what value of y

$$|E|\leqslant\tfrac{1}{2}=\frac{6}{7(3)^{1/2}}$$

or just over $\frac{1}{2}$%.

COMPARISON WITH OTHER METHODS

In previous work[2] with a non-Newtonian solution of polyisobutylene (a 2 g per litre solution of Oppanol B200 in decalin), parallel disc data treated on the basis of equation (4) showed excellent agreement with cone and plate measurements. In *Figure 2* the results from equation (4) are compared with data obtained with the simple 4/3 approximation. Over the range of measurements the solution approximates to a power law liquid with power 0.42 and this means that the error should be about 1%. This is near the value of n for maximum error. The agreement shown in *Figure 2* is quite satisfactory.

SECOND ORDER METHOD

We have seen that with the Casson equation it is not possible to choose a value of λ which enables equation (7) to be satisfied for all values of y. Indeed this is true when $T(y)$ takes the form

$$T(y)=\alpha+\beta y+\gamma y^2$$

since then equation (7) leads to

$$\alpha+\tfrac{4}{3}\beta y+\tfrac{5}{3}\gamma y^2=\alpha+\beta\lambda y+\gamma\lambda^2y^2$$

If we take λ as 4/3 the quadratic term does not cancel. Thus we are led to try the approximate form

$$T+\frac{y}{3}\frac{\mathrm{d}T}{\mathrm{d}y}=pT(Hy)+qT(Ky) \quad (11)$$

where p, q, H and K are to be found. We find that these constants must satisfy

$$p+q=1$$
$$pH+qK=4/3$$
$$pH^2+qK^2=5/3$$

if a quadratic form of T is to satisfy equation (11) identically. Since we have three equations in four unknowns, at first sight we may add the equation

$$pH^3+qK^3=2$$

which would mean that a cubic equation for T would

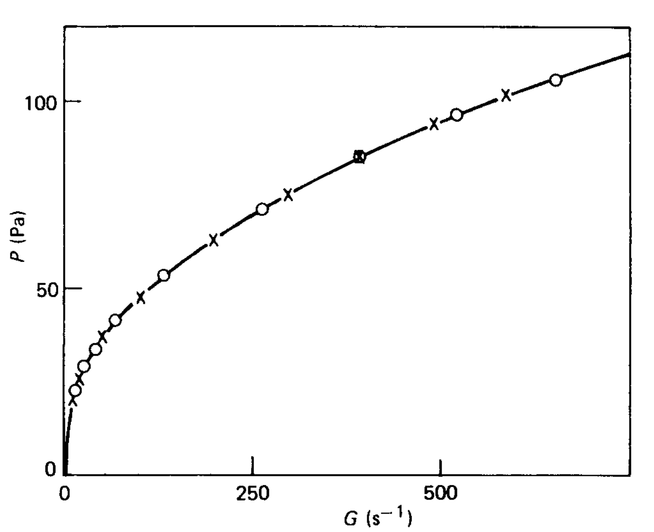

Figure 2 Flow curve for PIB solution: (○) results from equation (4); (×) first order approximation, $\lambda=4/3$

Table 1 Second order method

ω	H	K	p	q	$L=pH^3+qK^3$
(a) 6.0311	1.1328	0.7793	1.5674	−0.5674	2.00976
(b) 27/2	16/15	11/12	25/9	−16/9	2.00185

Table 2 E *versus* n when $T(y)=Ay^n$

n	1st order	(a)	(b)
0	0	0	0
0.2	-7.00×10^{-3}	4.77×10^{-4}	8.46×10^{-5}
0.4	-1.00×10^{-2}	5.94×10^{-4}	1.06×10^{-4}
0.6	-9.67×10^{-3}	4.88×10^{-4}	8.75×10^{-5}
0.8	-6.22×10^{-3}	2.62×10^{-4}	4.73×10^{-5}
1.0	0	0	0
1.2	8.78×10^{-3}	-2.34×10^{-4}	-4.27×10^{-5}
1.4	2.00×10^{-2}	-3.89×10^{-4}	-7.13×10^{-5}
1.6	3.34×10^{-2}	-4.23×10^{-4}	-7.78×10^{-5}
1.8	5.00×10^{-2}	-3.02×10^{-4}	-5.59×10^{-5}
2.0	6.67×10^{-2}	0	0
2.2	8.64×10^{-2}	5.06×10^{-4}	9.44×10^{-5}
2.4	1.08×10^{-1}	1.23×10^{-3}	2.31×10^{-4}
2.6	1.32×10^{-1}	2.20×10^{-3}	4.13×10^{-4}
2.8	1.58×10^{-1}	3.41×10^{-3}	6.44×10^{-4}
3.0	1.85×10^{-1}	4.88×10^{-3}	9.26×10^{-4}

satisfy equation (7) identically. However, it is not possible to find real values of the constants to satisfy the four equations. We must be content with three equations and some reasonable choice of the constants. It can be shown that if w is a parameter then the choice

$$p=(w+1.5)^2/6w$$
$$q=(w-1.5)^2/6w$$
$$H=(w+2.5)/(w+1.5)$$
$$K=(w-2.5)/(w-1.5)$$

satisfies the three equations. We now have to make a choice of w. Suppose we let

$$K=1/H^2$$

then w satisfies

$$4w^2-16w-49=0$$

giving

$$w=\tfrac{1}{2}[4+(65)^{1/2}]=6.03113 \text{ (choice (a))}$$

leading to

$$H=1.13278$$

Another choice for w is 27/2 (choice (b)). These choices are displayed in *Table 1*. The ideas behind the two choices are different. First, we notice that, in the last column, L gives measure of the accuracy achieved. If a cubic $T(y)$ is to satisfy equation (7) identically, then L has to be exactly two. Choice (b) was made to achieve a value of L as close to two as possible with rational and simple values of H and K. Choice (a) was made so that the values of $T(y)$ required to construct the flow curve would be uniformly spaced on a T *versus* $\log(y)$ curve. Thus, if

$$S_p=\log y+p\log(1.13278)$$

and T_p is the value of T corresponding to S_p then T_{p-2} and T_{p+1} are required to find the shear stress when the shear rate at the rim is Ry. The experimental information is then used economically. We now test the two methods for accuracy by using two techniques.

The power law liquid

If we take $T(y)$ in the form

$$T(y)=\alpha y^n$$

we find that the error E is not a function of y but only of n. We find

$$E=3(pH^n+qK^n)/(n+3)-1$$

Table 2 shows the values of E for the two choices of constants (a) and (b). We see that over a range of values of n from 0 to 2 the greatest error in either system is less than 0.06%. This is very likely to be very much less than the experimental error.

The sine function

We look at a form of T which makes the error a function of y. We have chosen

$$T(y)=\sin y$$

over the range $0<y<90$ since this should be a stringent test for the method. *Table 3* shows the error E for various values of y. We note again the small error.

ALTERNATIVE FORM OF FIRST ORDER TREATMENT

We have seen that if we use a parallel disc viscometer with radius R, gap c and angular velocity Ω, the torque T can be used to calculate a shear stress given by

$$P_a=3T/(2\pi R^3)$$

and this shear stress corresponds to a shear rate

$$G_a=\frac{3}{4}\cdot\frac{\Omega R}{c}$$

We can give an alternative physical description of this process. Suppose we measured T and we thought we were

Table 3 E *versus* y when $T(y)=\sin y$

y (degrees)	1st order	(a)	(b)
10	-1.41×10^{-3}	-3.70×10^{-5}	-7.00×10^{-6}
20	-5.67×10^{-3}	-1.45×10^{-4}	-2.73×10^{-5}
30	-1.28×10^{-2}	-3.13×10^{-4}	-5.90×10^{-5}
40	-2.31×10^{-2}	-5.23×10^{-4}	-9.82×10^{-5}
50	-3.65×10^{-2}	-7.45×10^{-4}	-1.39×10^{-4}
60	-5.36×10^{-2}	-9.32×10^{-4}	-1.72×10^{-4}
70	-7.48×10^{-2}	-1.01×10^{-3}	-1.81×10^{-4}
80	-1.01×10^{-1}	-8.45×10^{-4}	-1.41×10^{-4}
90	-1.34×10^{-1}	-2.17×10^{-4}	-7.82×10^{-6}

using a cone and plate viscometer. We would say that the shear stress was $3T/(2\pi R^3)$ and the shear rate $R\Omega/c$. However, the instrument is a parallel disc and the shear rate at any given radius for a parallel disc is less than the corresponding shear rate for a cone and plate. Thus, $3T(2\pi R^3)$ must correspond to a smaller shear rate, $v\Omega R/c$. Thus using

$$T+\frac{y}{3}\frac{\mathrm{d}T}{\mathrm{d}y}=P_{\mathrm{R}}\cdot\frac{2\pi R^3}{3}$$

we see that if we were to use the viscometer with the rim shear rate $v\Omega R/c$ then P_{R} would equal $3T/(2\pi R^3)$ and the left hand side would be evaluated with y replaced by vy. That is

$$\left(T+\frac{y}{3}\frac{\mathrm{d}T}{\mathrm{d}y}\right)_{vy}=T(y)$$

and now, carrying out the mathematical process of replacing y by y/v, we have

$$T+\frac{y}{3}\frac{\mathrm{d}T}{\mathrm{d}y}=T\left(\frac{y}{v}\right)$$

which is equation (7) with $\lambda=1/v$. We have already seen why we choose $v=3/4$.

Let us now follow another physical approximation idea. We again measure the torque T in our parallel disc system with Ω, c and R as the variables. This time we know that it is a parallel disc viscometer. Unfortunately, we think the liquid is Newtonian so we calculate an apparent viscosity η' given by

$$\eta'=2Tc/(\pi R^4\Omega)$$

but the liquid is non-Newtonian. The shear rate varies from zero to the rim value so the viscosity will vary from the centre to the rim. Somewhere in the gap the viscosity must be equal to η'. Let this position be vR. Then at this point the shear rate is

$$v\Omega R/c$$

and the shear stress is

$$\eta' v\Omega R/c=\frac{2Tc}{\pi R^4\Omega}\cdot\frac{v\Omega R}{c}=\frac{4}{3}vT\Big/\left(\frac{2}{3}\pi R^3\right)$$

Now, again let us rotate the disc so that the rim shear rate is $v\Omega R/c$. The shear stress at the rim is then

$$\frac{4}{3}vT\Big/\left(\frac{2}{3}\pi R^3\right)$$

The formula

$$T+\frac{y}{3}\frac{\mathrm{d}T}{\mathrm{d}y}=P_{\mathrm{R}}\frac{2}{3}\pi R^3$$

then tells us that

$$\left(T+\frac{y}{3}\frac{\mathrm{d}T}{\mathrm{d}y}\right)_{yv}=\frac{4}{3}\cdot vT(y)$$

and replacing y by y/v we have

$$T+\frac{y}{3}\frac{\mathrm{d}T}{\mathrm{d}y}=\frac{4}{3}vT(y/v)$$

Or with $\lambda=1/v$

$$T+\frac{y}{3}\frac{\mathrm{d}T}{\mathrm{d}y}=\frac{4}{3}\cdot\frac{1}{\lambda}T(\lambda y) \qquad (12)$$

we have an alternative first order theory. In effect this uses the simple Newtonian equations for G and P at a radius R/λ. At $\lambda=4/3$, that is at radius $3R/4$, the equations are identical to those of the original first order theory. We may investigate the choice of λ which gives the best fit with particular stress/shear rate relations. In some ways it is to be preferred to the original first order theory. It is, however, not easy to see a natural extension to higher order theories.

OTHER SYSTEMS

The problem of evaluating the derivative of an experimentally defined function is a common one in rheology and elsewhere. It may be that the ideas outlined in this paper will have other applications. We mention one such problem in rheology which appears to be almost identical to the current one.

The flow of a non-Newtonian liquid through a circular tube[5] leads to the equation

$$\pi R^3 G=\frac{1}{s^2}\frac{d}{ds}(s^3 Q) \qquad (13)$$

where R is the radius of the tube, G and s are the shear rate and the shear stress at the wall and Q is the rate of flow. Q and s are easily measured since s is given by

$$\pi R^2 s=\mathrm{d}P/\mathrm{d}L$$

where $\mathrm{d}P/\mathrm{d}L$ is the presssure gradient driving the flow. We may rearrange the equation in the form

$$Q+\frac{s}{3}\frac{\mathrm{d}Q}{\mathrm{d}s}=\frac{\pi R^3}{3}G \qquad (14)$$

This, apart from a trivial change to the right hand side, is the equation we have been manipulating. It is therefore possible to process tube flow data in the manner of this paper to calculate non-Newtonian viscosities.

CONCLUSIONS

An approximation method has been developed which enables viscosity/shear rate data to be obtained with a parallel disc viscometer in the same way as with a cone and plate instrument. It has been shown that the errors introduced by using the first order method should be less than 1%. For most practical purposes this should be adequate. Should a higher accuracy be required then either of the second order methods will reduce the error by an order of magnitude. Which second order method is used will depend on the arrangements for gap setting and angular velocity specification on the particular instrument used.

With the different techniques available, parallel plate geometry may be considered as a possible alternative to cone and plate even in situations which do not demand large plate separations. With both c and Ω as experimental variables a very wide range of shear rates can be achieved while errors associated with gap setting and with geometric imperfections are likely to be very small. Cheng[4] has found that commercially available cones show significant departure from true conical form and it is evident that a flat disc can be produced with greater precision than a cone.

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